SOME MODELS OF MICROPOLAR VISCOELASTIC MEDIA

A. T. Listrov and Yu. A. Shurinov

Models of viscoelastic micropolar media are constructed which generalize the Poynting–Thomson, Jeffreys–Lesersic, and Burgers–Mindlin models [2, 3] for the case where the medium has microinertia.

Generalized micropolar Maxwell and Voigt models are considered in [1]. The models discussed can be used for describing the mechanical behavior of suspensions, colloidal solutions, concretes, etc. The propagation of small shear disturbance in an unbounded medium is investigated.

Graphs are plotted of the propagation velocity and damping factors of periodic waves as a function of the frequency of the disturbances for each of the models under consideration. The calculations were performed on the M-220 computer.

The results obtained permit making a number of conclusions concerning the effect of relaxation and the elastic aftereffect of the media on the propagation velocity and damping of shear waves in the presence of microinertia and moment stresses.

1. The theory of linear micropolar viscoelasticity is based on the following equations of conservation of mass, change of momentum, moment of momentum, and energy for micropolar media [1]:

$$\begin{gathered}\rho^{\cdot} + (\rho v_k)_{,k} = 0, \qquad t_{kl,k} + \rho(f_l + v_l^{\cdot}) = 0 \\ m_{rk,r} + \varepsilon_{klr} t_{lr} + \rho(l_k - j\nu_k^{\cdot}) = 0 \qquad (k, l, r = 1, 2, 3) \\ \rho e^{\cdot} = t_{kl} d_{lk} + \varepsilon_{klr} t_{kl}(\omega_r - \nu_r) + m_{kl}\nu_{l,k} + q_{k,k} + \rho h\end{gathered} \tag{1.1}$$

Here ρ is the mass density, v_k is the velocity vector of a point of the continuum, ω_k is a vector characterizing the average angular velocity of rotation of the particles composing a point of the continuum, ν_k is the velocity vector of microrotation of the particle, f_k is the mass force vector, q_k is the heat flux vector, l_k is the mass moment vector, h is the heat source, e is the internal specific energy, j is the average value of the moment of inertia, t_{kl} and m_{kl} are the force and moment stress tensors, respectively, and ε_{klr} is the unit pseudotensor.

In the linear theory the dot over the index denotes a partial derivative with respect to time. The vectors of displacements u_k and microrotation φ_k satisfy the following kinematic relations

$$v_k = u_k^{\cdot}, \quad \nu_k = \varphi_k^{\cdot}, \quad 2\omega_k = \varepsilon_{klr} v_{r,l}, \quad 2d_{kl} = (v_{k,l} + v_{l,k})$$

The determining rheological equations of micropolar viscoelasticity for the deviatoric and spherical parts of the tensors in an operator form [1] are

$$\begin{gathered}\mathbf{P}p = \mathbf{Q}\varepsilon, \quad \mathbf{R}t_{kl}^{\circ} = \mathbf{S}\varepsilon_{kl}^{\circ} + \mathbf{T}\varepsilon_{lk}^{\circ}, \quad t_{kl} = -p\delta_{kl} + t_{kl}^{\circ} \\ \mathbf{P}'m = \mathbf{Q}'\varphi, \quad \mathbf{R}'m_{kl}^{\circ} = \mathbf{S}'\overset{\circ}{\varphi}_{k,l} + T'\overset{\circ}{\varphi}_{k,l}, \quad m_{kl} = m\delta_{kl} + m_{kl}^{\circ} \\ \varepsilon_{kl} = -\varepsilon\delta_{kl} + \varepsilon_{kl}^{\circ}, \quad 3p = -t_{rr}, \quad 3m = m_{rr} \\ \varphi_{k,l} = \varphi\delta_{kl} + \overset{\circ}{\varphi}_{k,l}, \quad 3\varepsilon = -\varepsilon_{rr}, \quad 3\varphi = \varphi_{r,r}, \quad \varepsilon_{kl} = u_{l,k} - \varepsilon_{klr}\varphi_r\end{gathered} \tag{1.2}$$

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Here $\mathbf{P}, \mathbf{P}', \mathbf{Q}, \mathbf{Q}', \mathbf{R}, \mathbf{R}', \mathrm{S}, \mathrm{S}', \mathbf{T}, \mathbf{T}'$ are differential operators of the form

$$\mathbf{P} \equiv \sum_{k=0}^{n} p_k \frac{\partial^k}{\partial t^k}, \quad \mathbf{P}' \equiv \sum_{k=0}^{n} p_k' \frac{\partial^k}{\partial t^k}, \ldots, \quad \mathbf{T}' \equiv \sum_{k=0}^{n} t_k' \frac{\partial^k}{\partial t^k} \tag{1.3}$$

The operator equations for the force and moment stresses follow from (1.2)

$$\begin{gathered} \mathbf{R} t_{kl} + {}^1/_3 (\mathbf{P} - \mathbf{R}) t_{rr} \delta_{kl} = {}^1/_3 (\mathbf{Q} - \mathrm{S} - \mathbf{T}) \varepsilon_{rr} \delta_{kl} + \mathrm{S} \varepsilon_{kl} + \mathbf{T} \varepsilon_{lk} \\ \mathbf{R}' m_{kl} + {}^1/_3 (\mathbf{P}' - \mathbf{R}') m_{rr} \delta_{kl} = {}^1/_3 (\mathbf{Q}' - \mathrm{S}' - \mathbf{T}') \varphi_{r,r} \delta_{kl} + \mathrm{S}' \varphi_{k,l} + \mathbf{T}' \varphi_{l,k} \end{gathered} \tag{1.4}$$

We introduce the following notations for the rheological models being considered [3]: N is the Newton model, K is the Kelvin–Voigt, M the Maxwell, P the Poynting–Thomson, J the Jeffreys–Lesersic, and B the Burgers–Mindlin.

The generalized models, taking into account microinertia, asymmetry of the force stress tensor, and the presence of moment stresses, will be denoted by the same letters with an asterisk, i.e., N*, ..., B*.

The determining equations of the models N, N* ,..., B, B* are obtained from the general operator equations (1.4). For this purpose it suffices to limit ourselves to $n = 2$ in (1.3) and to make appropriate assumptions concerning the values of the coefficients p_k, p_k', ..., t_k'.

In (1.3) we set

$$n = 2, \quad p_1 = p_1' = r_0 = r_0' = 1,$$
$$p_0 = p_0' = p_2 = p_2' = q_0 = q_0' = q_2 = q_2' = 0$$

Under these assumptions Eqs. (1.4) will take the form

$$\begin{gathered} -\frac{1}{3}\left[1 + (r_1 - 1)\frac{\partial}{\partial t} + r_2 \frac{\partial^2}{\partial t^2}\right] t_{rr}\delta_{kl} + \left(1 + r_1 \frac{\partial}{\partial t} + r_2 \frac{\partial^2}{\partial t^2}\right) t_{kl} \\ = -\frac{1}{3}(s_2 + t_2)\ddot{\varepsilon}_{rr}\delta_{kl} + s_2 \ddot{\varepsilon}_{kl} + t_2 \ddot{\varepsilon}_{lk} - \frac{1}{3}(s_1 - t_1 - q_1)\dot{\varepsilon}_{rr}\delta_{kl} \\ + s_1 \dot{\varepsilon}_{kl} + t_1 \dot{\varepsilon}_{lk} - \frac{1}{3}(s_0 + t_0)\varepsilon_{rr}\delta_{kl} + s_0 \varepsilon_{kl} + t_0 \varepsilon_{lk} \\ \dot{m}_{kk} = q' \dot{\varphi}_{k,k} \\ -\frac{1}{3}\left[1 + (r_1' - 1)\frac{\partial}{\partial t} + r_2' \frac{\partial^2}{\partial t^2}\right] m_{rr}\delta_{kl} + \left(1 + r_1' \frac{\partial}{\partial t} + r_2' \frac{\partial^2}{\partial t^2}\right) m_{kl} \\ = -\frac{1}{3}(s_2' + t_2')\ddot{\varphi}_{r,r}\delta_{kl} + s_2' \ddot{\varphi}_{k,l} + t_2' \ddot{\varphi}_{l,k} - \frac{1}{3}(s_1' - t_1' - q_1')\dot{\varphi}_{r,r}\delta_{kl} \\ + s_1' \dot{\varphi}_{k,l} + t_1' \dot{\varphi}_{l,k} - \frac{1}{3}(s_0' + t_0')\varphi_{r,r}\delta_{kl} + s_0' \varphi_{k,l} + t_0' \varphi_{l,k}, \quad \dot{t}_{kk} = q_1 \dot{\varepsilon}_{kk}, \\ \dot{m}_{kk} = q_1' \dot{\varphi}_{kk} \end{gathered} \tag{1.5}$$

Here r_1, r_1', ..., t_2, t_2' are constant coefficients characterizing the elastic and viscous properties of the media.

Relationships (1.5) contain all determining equations of the models considered.

The determining equations for any of the models N, N*, ..., B, B* can be obtained formally from (1.5) by means of Table 1.

In the row of this table the plus signs indicate those coefficients which should be retained in Eqs. (1.5) for the given model.

In this case, for models N, K, M, P, J, and B the nonzero coefficients next to each other in the table should be considered equal and, furthermore, we set $\varphi_r \equiv 0$ in (1.2).

2. We will consider periodic shear waves in an unbounded medium

$$\begin{gathered} \mathbf{u} = [u_1(x_2, t), 0, 0], \quad \boldsymbol{\varphi} = [0, 0, \varphi_3(x_2, t)], \quad f_k = l_k = 0 \\ u_1 = u_1^0 \exp[i(kx_2 + \omega t)], \qquad k = \omega / c + i\xi \end{gathered}$$

TABLE 1

	r_1	r'_1	r_2	r'_2	s_0	t_0	s'_0	t'_0	s_1	t_1	s'_1	t'_1	s_2	t_2	s'_2	t'_2	q_1	q'_1
N^*									+	+							+	+
N									+	+							+	
K^*					+	+	+	+	+	+	+	+					+	+
K					+	+			+	+							+	
M^*	+	+							+	+	+	+					+	+
M	+								+	+							+	
P^*	+	+			+	+	+	+	+	+	+	+					+	+
P	+				+	+			+	+							+	
J^*	+	+							+	+	+	+	+	+	+	+	+	+
J	+								+	+			+	+			+	
B^*	+	+	+	+					+	+	+	+	+	+	+	+	+	+
B	+		+						+	+			+	+			+	

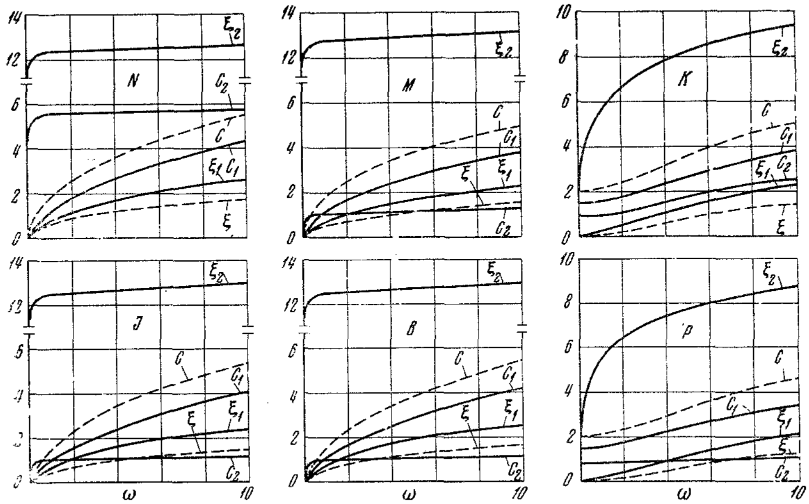

Fig. 1

Here k is the wave number, c is the propagative velocity of the wave, ξ is the damping factor, and ω is the angular frequency.

The dispersion equation for disturbances in micropolar media has the form

$$\begin{gathered} A_1A_3A_5k^4 + (i\rho\omega^2A_1^2A_5 - iA_2A_4A_6 - iA_3A_4A_6 \\ + i\rho j\omega^2A_1A_3A_6)\,k^2 + 2\rho\omega^2A_1A_4A_6 - \rho^2 j\omega^4A_1^2A_6 = 0, \\ A_6 = 1 - i\omega r_1' - \omega^2 r_2' \\ A_1 = 1 - i\omega r_1 - \omega^2 r_2, \quad A_2 = -\omega t_1 + i\,(\omega^2 t_2 - t_0), \\ A_3 = -\omega s_1 + i\,(\omega^2 s_2 - s_0) \\ A_4 = \omega^2\,(t_2 - s_2) - (t_0 - s_0) + i\omega\,(t_1 - s_1), \\ A_5 = -\omega t_1' + i\,(\omega^2 t_2' - t_0') \end{gathered} \tag{2.1}$$

The following values of the coefficients were used in the SI system in the numerical solution of Eq. (2.1) on the M-220 computer:

$$\begin{gathered} r_1 = 2\cdot10^{-2}, \quad r_2' = 10^{-4}, \quad s_0 = 4, \quad s_1 = 1.6, \quad t_2 = 10^{-3}, \quad \rho = 1 \\ r_2 = t_2' = 10^{-5}, \quad t_0 = t_1' = s_2 = 10^{-2}, \quad t_0' = t_1 = r_1' = 10^{-1}, \\ j = 5\cdot10^{-2} \end{gathered}$$

The graphs of the functions $c(\omega)$ and $\xi(\omega)$ for models N, N*, ..., B, B* are given in Fig. 1.

The values of ω are indicated on the x axis and the values of the damping factors and propagation velocities of the shear waves on the y axis.

The propagation velocity and damping factor of an ordinary shear wave for models N, K, M, P, J, and B are denoted in the graphs respectively by the letters c and ξ.

We note that two shear waves exist for micropolar media [4]: an ordinary shear wave which has a propagation velocity c_1 and damping factor ξ_1 and a shear wave due to the presence of moment stresses. The latter has propagation velocity c_2 and damping factor ξ_2.

The graphs of $c(\omega)$, $\xi(\omega)$ for model N and graphs of $c_1(\omega)$, $\xi_1(\omega)$, $c_2(\omega)$, $\xi_2(\omega)$ for the generalized model N* are shown in the figure with the letter N. The graphs with the letters M, J, B, K, and P refer respectively to the ordinary and generalized models of Maxwell, Jeffreys–Leversic, Burgers–Mindlin, Kelvin–Voigt, and Poynting–Thomson.

We see from Fig. 1 that for all the micropolar media the second shear wave has a large damping factor ξ_2.

It follows from the figure with letter N that in micropolar fluid N* the propagation velocity of the shear wave c_2 in the frequency range considered is greater than the propagation velocity c_1 of the ordinary shear wave.

We see from a comparison of the graphs that the presence of such properties as stress relaxation and elastic aftereffect for media M*, J*, B*, and K* is related to a substantial decrease of the propagation velocity of wave c_2. In this case both the elastic aftereffect and stress relaxation of the media have little effect on the damping factor ξ_2.

We note also that for all models consideration of microinertia and moment stresses leads to some decrease of the propagation velocity of the ordinary shear wave and to an increase of its damping factor.

LITERATURE CITED

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